

# Tucker type lemmas for $G$ -spaces

Oleg R. Musin\* and Alexey Yu. Volovikov

## Abstract

The classical Tucker lemma is a combinatorial version of the Borsuk–Ulam theorem. In this paper we consider several generalizations of this theorem for  $G$ -spaces that yield Tucker’s type lemmas for  $G$ -simplicial complexes and manifolds.

**Keywords:** Borsuk–Ulam theorem, Tucker’s lemma,  $G$ -space,  $G$ -index

## 1 Introduction

The classical Borsuk–Ulam theorem states that *for any continuous mapping  $f : \mathbb{S}^d \rightarrow \mathbb{R}^d$  there is a point  $x \in \mathbb{S}^d$  such that  $f(-x) = f(x)$*  [2].

The Tucker [22] and Ky Fan [8] lemmas are discrete analogs of the Borsuk–Ulam theorem. Let  $T$  be some triangulation of the  $d$ -dimensional ball  $\mathbb{B}^d$ . We call  $T$  *antipodally symmetric on the boundary* if the set of simplices of  $T$  contained in the boundary  $\partial \mathbb{B}^d = \mathbb{S}^{d-1}$  of the ball  $\mathbb{B}^d$  is an antipodally symmetric triangulation of  $\mathbb{S}^{d-1}$ , that is if  $s \subset \mathbb{S}^{d-1}$  is a simplex of  $T$ , then  $-s$  is also a simplex of  $T$ .

**Theorem 1.1. (Tucker’s lemma)** *Let  $T$  be a triangulation of  $\mathbb{B}^d$  that is antipodally symmetric on the boundary. Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be a labeling of the vertices of  $T$  that satisfies  $L(-v) = -L(v)$  for every vertex  $v$  on the boundary. Then there exists an edge in  $T$  that is **complementary**, i.e. its two vertices are labeled by opposite numbers.*

Consider also the following version of Tucker’s lemma:

**Theorem 1.2.** *Let  $T$  be a centrally symmetric triangulation of the sphere  $\mathbb{S}^d$ . Let*

$$L : V(T) \rightarrow \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be an equivariant labeling, i.e.  $L(-v) = -L(v)$ . Then there exists a complementary edge.*

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It is well known, see [12], that these theorems are equivalent to the Borsuk–Ulam theorem. In particular the first theorem is equivalent to the following version of Borsuk–Ulam theorem:

**Theorem 1.3.** *For any continuous mapping  $f : \mathbb{B}^d \rightarrow \mathbb{R}^d$  such that  $f$  is odd on the boundary  $\partial \mathbb{B}^d = \mathbb{S}^{d-1}$ , there exists a point  $x \in \mathbb{B}^d$  such that  $f(x) = 0 \in \mathbb{R}^d$ .*

In [14] it was shown that similar statement holds in a case when  $\mathbb{S}^{d-1} = \partial M^d$  where  $M^d$  is a manifold. In [18] we extended this result for more general spaces.

In this paper we prove  $G$ -analogs of the above results for a finite group  $G$ .

Let  $X$  be a free  $G$ -space. In Section 2 we define the *topological index*  $\text{t-ind}^G X$ . The main result of Section 3 is the following extension of Tucker’s lemma:

**Theorem 3.1.** *Let  $X$  be a simplicial complex with a free simplicial  $G$ -action. Then  $\text{t-ind}^G X \geq n$  if and only if for any equivariant  $(G, n)$ -labeling of the vertex set of an arbitrary equivariant triangulation of  $X$  there exists a complementary edge.*

In Section 4 we consider the *cohomological index*  $\text{ind}^G X$  and its properties. In Section 5 we use this notion to obtain  $G$ -generalizations of the Borsuk–Ulam theorem. In particular, Theorems 5.1–5.3 for  $G$ -spaces imply the following result for manifolds:

**Theorem 5.4.** *Let  $M^n$  be a compact connected orientable manifold (or a pseudomanifold) with the connected boundary  $\partial M$ , and assume that  $G$  can act freely on  $\partial M$ . Consider a continuous mapping  $f : M \rightarrow \mathbb{R}^n$  such that  $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$  is an equivariant map, where  $\mathbb{R}^n$  is considered as a semifree  $G$ -space with the unique fixed point in the origin  $0 \in \mathbb{R}^n$ . If  $\text{ind}^G \partial M = n - 1$  then the zero set  $Z_f = f^{-1}(0)$  is not empty.*

Assume that  $G$  can act freely on  $\mathbb{S}^{d-1}$ . Then there is an obvious semi-free  $G$ -action on  $\mathbb{R}^d$  with the unique fixed point  $0 \in \mathbb{R}^d$  (and free on  $\mathbb{R}^d \setminus 0$ ). Note that the degree of any equivariant map  $\mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$  equals 1 modulo  $|G|$  (see [9]). Actually, it implies that for any continuous map  $f : \mathbb{B}^d \rightarrow \mathbb{R}^d$  that is equivariant on the boundary the zero set  $f^{-1}(0)$  is not empty. Theorem 5.5 extends this fact for a case when  $\mathbb{S}^{d-1}$  is embedded to a space  $X$ .

In Section 6 we discuss an alternative approach for  $G$ -versions of the Borsuk–Ulam theorem. Namely, we prove Bourgin–Yang type theorems for  $G$ -spaces.

In Section 7 we consider Tucker type lemmas for bounded spaces. In particular, Theorem 6.1 yields the following theorem for manifolds:

**Theorem 7.2.** *Let  $M^n$  be a compact PL manifold with boundary. Suppose that  $\partial M$  is homeomorphic to the sphere  $\mathbb{S}^{n-1}$  and there exists a free PL action of a group  $G$  on  $\partial M \approx \mathbb{S}^{n-1}$ . Then for any  $(G, n)$ -labeling of the vertex set of an arbitrary triangulation of  $M$  that is an equivariant on the boundary there exists a complementary edge.*

## 2 Topological index

Consider a group  $G$  as a discrete free  $G$ -space. Let  $J^m(G) = G * \cdots * G$  be the join of  $m$ -copies of  $G$  with the diagonal action of  $G$ .

**Definition 2.1.** Let  $X$  be a free  $G$ -space. Topological index  $\text{t-ind}^G X$  equals minimal  $n$  such that there exists an equivariant map  $X \rightarrow J^{n+1}(G)$ . If no such  $n$  exists, then  $\text{t-ind}^G X = \infty$ .

**Remark 2.1.** 1) We can take  $E_G = G * \dots * G * \dots = J^\infty(G)$  as a total space of the universal  $G$ -bundle  $E_G \rightarrow B_G$ .

2) If  $G = \mathbb{Z}_2$  then  $J^{m+1}(\mathbb{Z}_2)$  is equivariantly homeomorphic to  $S^m$ , since  $SY = Y * \mathbb{Z}_2$ , where  $SY$  is the suspension, and

$$S^m = SS^{m-1} = S^{m-1} * \mathbb{Z}_2 = S^{m-2} * \mathbb{Z}_2 * \mathbb{Z}_2 = \dots = J^{m+1}(\mathbb{Z}_2).$$

3) For a cyclic group  $G = \mathbb{Z}_q$  we can take in the definition of index the following sequence of test spaces:  $G, S^1, S^1 * G, S^3, S^3 * G, S^5, S^5 * G, \dots$ , where each odd dimensional sphere is considered with some free action of  $G = \mathbb{Z}_q$  (recall that there exists no free action on even dimensional sphere of a group with odd order divisor).

The main property of the topological index:

*If  $X \rightarrow Y$  is equivariant then  $\text{t-ind}^G X \leq \text{t-ind}^G Y$ .*

It is not hard to see that if  $X$  is either compact, or paracompact and finite-dimensional, then  $\text{t-ind}^G X < \infty$ , in the second case  $\text{t-ind}^G X \leq \dim X$ . For the proof one can use nerves of  $G$ -invariant coverings of a  $G$ -space and the fact that  $J^{n+1}(G)$  is  $n$ -universal, i.e. any  $G$ -CW-complex of dimension not exceeding  $n$  can be mapped equivariantly to  $J^{n+1}(G)$ . The equality  $\text{t-ind}^G J^{n+1}(G) = n$  can be proved using (co)homological index (see next section), which is a lower bound for topological index. Proofs that don't use cohomological indexes can be found in [12].

For  $G = \mathbb{Z}_2$  this index was introduced by Yang [29] under the name  $B$ -index (Yang also introduced homological index which is discussed below). For finite groups topological index was introduced by M. Krasnosel'skii and in general case (for topological groups) by Albert Schwarz under the name genus (more precise genus is by 1 greater than topological index). In fact Schwarz [20] introduced and studied more general notion of genus of a fiber space which generalize the notions of the Lusternik–Shnirelman category and of Krasnosel'skiy genus of a covering (it is valid for a continuous surjective map).

On his web page Schwarz [21] writes: "The same notion was rediscovered (under another name) 25 years later by S. Smale who used to estimate topological complexity of algorithms".

Nowadays this notion is usually called sectional category.

### 3 Tucker type lemmas for $G$ -spaces

Let  $X$  be a simplicial complex and  $C$  be a finite set. Recall that a  $C$ -labeling (coloring) of  $X$  is a map  $V(X) \rightarrow C$  of the vertex set  $V(X)$  to  $C$ . For  $C = G \times \{1, \dots, n\}$  we say that we have a  $(G, n)$ -labeling. Thus a  $(G, n)$ -labeling prescribes to each vertex some pair  $(g, k)$  where  $g \in G$  and  $k \in \{1, \dots, n\}$ .

Now we define equivariant labelings.

**Definition 3.1.** Let  $X$  be a simplicial complex with a simplicial  $G$ -action, where  $G$  is a finite group, and  $C$  is a finite  $G$ -set. An equivariant  $C$ -labeling (coloring) of  $X$  is an equivariant map  $V(X) \rightarrow C$  of the vertex set  $V(X)$  to  $C$ . For  $C = G \times \{1, \dots, n\}$ , where  $G$  acts on the first factor by left multiplication and on the second factor the action is trivial, we call  $C$ -labeling as equivariant  $(G, n)$ -labeling.

**Definition 3.2.** An edge in  $X$  is called complementary if labels of its vertices belong to the same orbit in  $C$ . For  $(G, n)$ -labeling it means that vertices of a complementary edge have the form  $(g_1, k)$  and  $(g_2, k)$ ,  $g_1 \neq g_2$ , for some  $k \in \{1, \dots, n\}$ .

Obviously, for  $G = \mathbb{Z}_2$  an equivariant  $(\mathbb{Z}_2, n)$ -labeling is just the same as equivariant  $\{\pm 1, \dots, \pm n\}$ -labeling, and the sum of labels of a complementary edge (for  $(\mathbb{Z}_2, n)$ -labeled complex  $X$ ) equals zero.

**Theorem 3.1.**  $\text{t-ind}^G X \geq d$  if and only if for any equivariant  $(G, d)$ -labeling of the vertex set of an arbitrary equivariant triangulation of  $X$  there exists a complementary edge.

*Proof.* Assume there is a  $(G, d)$ -labeling of the vertex set of an equivariant triangulation of  $X$  without complementary edges. Then such a labeling provides an equivariant map  $X \rightarrow J^d(G)$ . This contradicts with the assumption that  $\text{t-ind}^G X \geq d$ .

Now assume that for any equivariant  $(G, d)$ -labeling of the vertex set of an arbitrary equivariant triangulation of  $X$  there exists a complementary edge. Assume that  $\text{t-ind}^G X < d$ . Then there exists an equivariant continuous map  $X \rightarrow J^d(G)$  and an equivariant simplicial approximation of this map which is a simplicial map of some triangulation of  $X$ . So there exists a  $(G, d)$ -labeling of  $J^d(G)$  without complementary edges. Thus, the inverse image of this labeling is a  $(G, d)$ -labeling of  $X$  without complementary edges, a contradiction.  $\square$

## 4 Cohomological index

Consider first the case of an action of the group  $\mathbb{Z}_p$  of prime order  $p$  (the case  $p = 2$  was considered in [18]). Using Smith's sequences we can define for a free  $\mathbb{Z}_p$ -space its cohomological index  $\text{ind}_p X \in \{0, 1, 2, \dots, \infty\}$  possessing the following properties (see [26] for details):

1. If there exists an equivariant map  $X \rightarrow Y$  of free  $\mathbb{Z}_p$ -spaces then  $\text{ind}_p(X) \leq \text{ind}_p(Y)$ .
2. If  $X = A \cup B$  are open invariant subspaces, then

$$\text{ind}_p(X) \leq \text{ind}_p(A) + \text{ind}_p(B) + 1.$$

3. Tautness: If  $Y$  is a closed invariant subspace of  $X$ , then there exists an open invariant neighborhood of  $Y$  such that  $\text{ind}_p(Y) = \text{ind}_p(U)$ .

4.  $\text{ind}_p(X) > 0$  if  $X$  is connected.
5. Let  $X$  be either compact, or paracompact and finite dimensional. Then  $\text{ind}_p(X) < \infty$ .
6. Assume that  $X$  is connected and  $H^i(X; \mathbb{Z}_p) = 0$  for  $0 < i < N$ . Then  $\text{ind}_p(X) \geq N$ .
7. Assume that  $X$  is finite dimensional and  $H^i(X; \mathbb{Z}_p) = 0$  for  $i > d$ . Then  $\text{ind}_p(X) \leq d$ .

8. If there exists an equivariant map  $f : X \rightarrow Y$  and  $\text{ind}_p(X) = \text{ind}_p(Y) = k < \infty$  then  $0 \neq f^* : H^k(Y; \mathbb{Z}_p) \rightarrow H^k(X; \mathbb{Z}_p)$ .

Here Čech cohomology groups are used.

For  $G = \mathbb{Z}_2$  this index was introduced by Yang [29]. In [30] Yang actually used this index for  $G = \mathbb{Z}_3$  without naming it. Conner & Floyd in [4] introduced for any finite group  $G$  and a commutative ring with unit  $L$  a cohomological index for which they used notation  $\text{co-ind}_L(\cdot)$ . At the same time A.S. Schwarz [20] introduced homological genus. It can be shown that homological genus equals  $\text{co-ind}_{\mathbb{Z}}(\cdot) + 1$ , and  $\text{co-ind}_{\mathbb{Z}_p}(\cdot)$  for  $G = \mathbb{Z}_p$  coincides with  $\text{ind}_p(\cdot)$ .

In what follows the property 8 will serve as our main tool.

For example, from properties 1 and 8 we immediately obtain that  $\text{ind}_p(\cdot)$  is stable, i.e.  $\text{ind}_p X * \mathbb{Z}_p = \text{ind}_p X + 1$  (see [26, Corollary 3.1]), so if  $\text{ind}_p X$  is finite then there exists no equivariant map  $X * \mathbb{Z}_p \rightarrow X$ . As a direct consequence we have the following assertion:

**Proposition 4.1.** *Let  $X$  be a free  $H$ -space, where  $X$  is either compact or paracompact and finite dimensional, and  $H$  is any topological group which has a nontrivial finite subgroup. Then there exists no equivariant map  $X * H \rightarrow X$ .*

The same result was obtained independently by Passer in [19]. Proposition 4.1 gives the partial solution to the following conjecture of Baum, Dabrowski and Hajac:

**Conjecture 4.1** ([1], Conjecture 2.2). *Let  $X$  be a compact Hausdorff space with a continuous free action of a nontrivial compact Hausdorff group  $G$ . Then, for the diagonal action of  $G$  on the join  $X * G$ , there does not exist an equivariant continuous map  $f : X * G \rightarrow X$ .*

For arbitrary nontrivial compact Hausdorff group the conjecture was proved by Chirvasitu and Passer in [3] and noncommutative case was considered in the paper of Dabrowski, Hajac and Neshveyev [6].

We give one more simple proof of proposition 4.1 below.

**Proposition 4.2.** *Let  $X$  and  $Y$  be free  $G$ -spaces and  $p$  is a prime divisor of order  $|G|$ . Assume that  $\tilde{H}^i(X; \mathbb{Z}_p) = 0$  for  $i \leq n$  and  $\text{t-ind}^G Y \leq n$ . Then there exists no equivariant map from  $X$  to  $Y$ . In particular no equivariant map exists if  $\dim Y \leq n$ .*

*Proof.* Obviously the problem reduces to the case  $G = \mathbb{Z}_p$ . Then the proposition follows from properties 1, 6 of the index  $\text{ind}_p(\cdot)$  and the fact that  $\text{ind}_p Y \leq \text{t-ind}^G Y \leq \dim Y$ , where  $Y$  is a free finite-dimensional  $\mathbb{Z}_p$ -space. However for the proof it is easier to use more simple index  $\text{in}_p(\cdot)$  which equals weak homological genus (introduced in [20]) minus 1.

To define  $\text{in}_p(X)$  for a paracompact free  $\mathbb{Z}_p$ -space  $X$  consider an equivariant map  $X \rightarrow J^\infty(\mathbb{Z}_p) = E_{\mathbb{Z}_p}$  and the map of factor-spaces  $\mu : X/\mathbb{Z}_p \rightarrow B_{\mathbb{Z}_p}$ . Recall that  $H^i(B_{\mathbb{Z}_p}; \mathbb{Z}_p) = \mathbb{Z}_p$ . Say that  $\text{in}_p(X) \geq n$  if  $\mu^* \neq 0$  in dimension  $n$ . It is easy to see that this assumption is equivalent to the assumption that  $\mu^* : H^i(B_{\mathbb{Z}_p}; \mathbb{Z}_p) \rightarrow H^i(X/\mathbb{Z}_p; \mathbb{Z}_p)$  is a monomorphism for  $i \leq n$ . To prove proposition 4.2 we need only to show that index  $\text{in}_p(\cdot)$  satisfies properties 1,

5 and 6. The most complicated property 6 follows easily from the consideration of the spectral sequence of a covering  $X \rightarrow X/\mathbb{Z}_p$  (from the spectral sequence of a bundle  $X \times_{\mathbb{Z}_p} E_{\mathbb{Z}_p} \rightarrow X/\mathbb{Z}_p$  with fiber  $X$ ).  $\square$

**Remark 4.1.** 1) Results like proposition 4.2 (generalizations can be obtained using [20, Theorem 17]) belong to A.S.Schwarz, since they follow trivially from [20, Theorem 17] and its corollaries 1 and 2 and properties of homological and weak homological genus introduced in [20].

2) A well known Dold's result [7] is a partial case of proposition 4.2.

3)  $\text{in}_p(\cdot)$  possesses all other properties except 2 and 8, and it can be shown that  $\text{in}_p(\cdot) \leq \text{ind}_p(\cdot)$  with the equality for  $p = 2$ .

4) The definition of  $\text{in}_p(\cdot)$  and its property 6 was rediscovered many times, see f.e. [11], [23, 24]. In [10] Liulevicius actually used this index without naming it.

*Second proof of proposition 4.1.* We argue by contradiction following the main line of the proof in [19]. Assume that there exists an equivariant map  $X * G \rightarrow X$ , then taking join with  $G$  we obtain an equivariant map  $X * G * G \rightarrow X * G$ , and hence a map  $X * G * G \rightarrow X$ . Iterating this procedure we obtain for any  $n$  an equivariant map  $X * J^n(G) \rightarrow X$ . We can assume that  $G = \mathbb{Z}_p$ . Then there exist equivariant maps  $J^{2m}(G) \rightarrow S^{2m-1}$  and  $S^{2m-1} \rightarrow J^{2m}(G)$ , so we have an equivariant map  $X * S^{2m-1} \rightarrow X$  for any  $m \in \mathbb{N}$ . From the assumptions it follows that  $\text{t-ind}^G X$  is finite. Taking  $m$  large enough we can make acyclicity of  $X * S^{2m-1}$  greater than  $\text{t-ind}^G X$  (use suspension isomorphism and the fact that the join with a sphere is the iterated unreduced suspension) and obtain a contradiction with proposition 4.2.  $\square$

Now we need to recall the definition of  $\text{ind}_p(\cdot)$ . Denote by  $\pi : X \rightarrow X/\mathbb{Z}_p$  the projection. Then there are two Smith sequences:

$$\dots \rightarrow H_p^k(X) \longrightarrow H^k(X) \xrightarrow{\pi^!} H^k(X/\mathbb{Z}_p) \xrightarrow{\delta_1} H_p^{k+1}(X) \rightarrow \dots$$

and

$$\dots \rightarrow H^k(X/\mathbb{Z}_p) \xrightarrow{\pi^*} H^k(X) \longrightarrow H_p^k(X) \xrightarrow{\delta_2} H^{k+1}(X/\mathbb{Z}_p) \rightarrow \dots$$

Here coefficients  $\mathbb{Z}_p$  are omitted and  $\pi^!$  is called the transfer.

Let us define  $s_{2d} : H^0(X/\mathbb{Z}_p) \rightarrow H^{2d}(X/\mathbb{Z}_p)$  and  $s_{2d+1} : H^0(X/\mathbb{Z}_p) \rightarrow H_p^{2d+1}(X)$  as  $s_{2d+1} = \delta_1 s_{2d}$  and  $s_{2d+2} = \delta_2 s_{2d+1}$  where  $s_0 = \text{id}$ , and put  $u_n(X) = s_n(1)$ ,  $1 \in H^0(X/\mathbb{Z}_p)$ . Then  $\text{ind}_p X$  equals maximal  $n$  such that  $u_n(X) \neq 0$ .

The following proposition is a partial converse to Property 8 (see also [25, Proposition 3.3]).

**Proposition 4.3.** *Let  $X$  and  $Y$  be free  $\mathbb{Z}_p$ -spaces and  $f : X \rightarrow Y$  an equivariant map. Assume that  $k$  is odd and*

- a)  $\text{ind}_p(Y) = k$ ,
- b)  $\dim X = k$ ,

- c)  $H^k(X; \mathbb{Z}_p) = H^k(Y; \mathbb{Z}_p) = \mathbb{Z}_p$ ,  
d)  $f^* : H^k(Y; \mathbb{Z}_p) \rightarrow H^k(X; \mathbb{Z}_p)$  is an isomorphism.  
Then  $\text{ind}_p(X) = k$ .

*Proof.* Put  $k = 2n + 1$ .

An equivariant map  $f : X \rightarrow Y$  between free  $\mathbb{Z}_p$ -spaces induces a map of factor spaces  $X/\mathbb{Z}_p \rightarrow Y/\mathbb{Z}_p$  and we have two commutative diagrams (for  $p$  odd) since Smith's sequences are functorial. Consider one of these diagrams:

$$\begin{array}{ccccccc} H^k(X/\mathbb{Z}_p) & \xrightarrow{\pi^*} & H^k(X) & \longrightarrow & H^k_\rho(X) & \xrightarrow{\delta_2} & H^{k+1}(X/\mathbb{Z}_p) \\ \uparrow & & \uparrow f^* & & \uparrow & & \uparrow \\ H^k(Y/\mathbb{Z}_p) & \longrightarrow & H^k(Y) & \longrightarrow & H^k_\rho(Y) & \xrightarrow{\delta_2} & H^{k+1}(Y/\mathbb{Z}_p) \end{array}$$

Since  $u_k(Y) \neq 0$  and  $\delta_2 u_k(Y) = 0$ , there is a nontrivial element  $\alpha \in H^k(Y) = \mathbb{Z}_p$  which is mapped onto  $u_k(Y)$ . Now  $u_k(Y)$  is mapped to  $u_k(X)$  and from assumption d) it follows that  $0 \neq f^* \alpha \in H^k(X) = \mathbb{Z}_p$  is mapped onto  $u_k(X)$ . Now we argue by contradiction. If  $u_k(X) = 0$  then  $H^k(X) \rightarrow H^k_\rho(X)$  is trivial. Since  $H^{k+1}(X/\mathbb{Z}_p) = 0$ , we obtain  $H^k_\rho(X) = 0$ . We have also  $H^{k+1}_\rho(X) = 0$ , since  $\dim X = k$ . From Smith's sequence

$$H^k_\rho(X) \longrightarrow H^k(X) \xrightarrow{\pi^!} H^k(X/\mathbb{Z}_p) \xrightarrow{\delta_1} H^{k+1}_\rho(X)$$

we see that  $\pi^!$  is an isomorphism and  $H^k(X/\mathbb{Z}_p) = \mathbb{Z}_p$ . From the first row of the above diagram it follows that  $\pi^* : H^k(X/\mathbb{Z}_p) \rightarrow H^k(X)$  is also an isomorphism, so  $\pi^! \circ \pi^*$  is an isomorphism, but this contradicts with the fact that  $\pi^! \circ \pi^*$  is the multiplication by  $p$ , i.e. zero homomorphism.  $\square$

Note that if  $X$  is a free  $\mathbb{Z}_p$ -space where  $p$  is an odd prime and  $\dim X = 2n + 1$ , then there exists an equivariant map  $f : X \rightarrow S^{2n+1}$ .

**Corollary 4.1.** *Let  $X$  be a free  $\mathbb{Z}_p$ -space where  $p$  is an odd prime. Assume that  $\dim X = 2n + 1$  and  $H^{2n+1}(X; \mathbb{Z}_p) = \mathbb{Z}_p$ , and denote by  $f : X \rightarrow S^{2n+1}$  an equivariant map. Then  $\text{ind}_p(X) = 2n + 1$  if and only if  $f^* : H^{2n+1}(S^{2n+1}; \mathbb{Z}_p) \rightarrow H^{2n+1}(X; \mathbb{Z}_p)$  is an isomorphism.*

In what follows we will use cohomological index with integer coefficients. This index is defined via homological genus introduced by Albert Schwarz in [20].

**Definition 4.1.** *Let  $X$  be a free  $G$ -space. We define  $\text{ind}^G X$ , the integer cohomological index of  $X$ , as its Schwarz's homological genus minus 1.*

**Remark 4.2.** 1) Using notation of Conner and Floyd [4] we have  $\text{ind}^G(\cdot) = \text{co-ind}_{\mathbb{Z}}(\cdot)$ .

2)  $\text{ind}^G(\cdot)$  is the largest cohomological index. In particular for  $G = \mathbb{Z}_p$  we have  $\text{ind}_p(\cdot) \leq \text{ind}^G(\cdot)$ . Also for any  $G$  we have  $\text{ind}^G(\cdot) \leq \text{t-ind}^G(\cdot)$ .

This cohomological index possesses similar properties:

1. If there exists an equivariant map  $X \rightarrow Y$  then  $\text{ind}^G(X) \leq \text{ind}^G(Y)$ .
2. If  $X = A \cup B$  are open invariant subspaces, then

$$\text{ind}^G(X) \leq \text{ind}^G(A) + \text{ind}^G(B) + 1.$$

3. Tautness: If  $Y$  is a closed invariant subspace of  $X$ , then there exists an open invariant neighborhood of  $Y$  such that  $\text{ind}^G(Y) = \text{ind}^G(U)$ .

4.  $\text{ind}^G(X) > 0$  if  $X$  is connected.

5. If  $X$  is either compact, or paracompact and finite dimensional then  $\text{ind}^G(X) < \infty$ .

6. Assume that  $X$  is connected and  $H^i(X; \mathbb{Z}) = 0$  for  $0 < i < N$ . Then  $\text{ind}^G(X) \geq N$ .

7. Assume that  $X$  is finite dimensional and  $H^i(X; \mathbb{Z}) = 0$  for  $i > d$ . Then  $\text{ind}^G(X) \leq d$ .

8. If there exists an equivariant map  $f : X \rightarrow Y$  and  $\text{ind}^G(X) = \text{ind}^G(Y) = k < \infty$  then  $0 \neq f^* : H^k(Y; \mathbb{Z}) \rightarrow H^k(X; \mathbb{Z})$ .

## 5 Borsuk–Ulam type theorems for bounded spaces

**Definition 5.1.** We say that  $h : X_0 \rightarrow X$  is  $n$ -cohomological trivial ( $n$ -c.t. map) over  $R$  if  $h^* : H^n(X; R) \rightarrow H^n(X_0; R)$  is the trivial homomorphism of cohomology groups with coefficients in  $R$  in dimension  $n$ . In the case when  $h$  is an embedding we call  $X_0$  an  $n$ -c.t.-subspace of  $X$  over  $R$ .

**Example 5.1.** Let  $X$  be a compact connected  $(n+1)$ -dimensional manifold with the connected boundary  $\partial X = X_0$ . Then  $X_0$  is an  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}_2$ , and if moreover  $X$  is orientable then  $X_0$  is an  $n$ -c.t.-subspace of  $X$  over  $R$  for any  $R$ .

Let a space  $X_0$  be a subspace of  $X$ . Denote by  $i : X_0 \rightarrow X$  the inclusion. Suppose  $X_0$  admits a free action of a finite group  $G$ . (Actually, we do not assume that  $X$  is a  $G$ -space.)

These assumptions on  $X$  and  $X_0$  will be used in what follows.

**Theorem 5.1.** Let  $Y$  be a  $G$ -space,  $Y_0$  its invariant closed subspace such that the action on  $Y \setminus Y_0$  is free, and  $f : X \rightarrow Y$  a continuous map. Assume that

- 1)  $n = \text{ind}^G X_0 = \text{ind}^G(Y \setminus Y_0)$ ,
  - 2)  $X_0$  is an  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}$ ,
  - 3)  $f|_{X_0} : X_0 \rightarrow Y$  is equivariant,
- then  $f^{-1}(Y_0) \neq \emptyset$ .

*Proof.* We argue by contradiction. If  $f^{-1}(Y_0) = \emptyset$  then  $f$  maps  $X$  into  $Y \setminus Y_0$  and  $f|_{X_0} : X_0 \rightarrow Y \setminus Y_0$  is equivariant. Since  $f|_{X_0} = f \circ i$  and  $i^*$  is trivial in dimension  $n$ , we obtain that  $(f|_{X_0})^* : H^n(Y \setminus Y_0; \mathbb{Z}) \rightarrow H^n(X_0; \mathbb{Z})$  is trivial, a contradiction with property 8 of index.  $\square$

Note that if  $\text{ind}^G(Y \setminus Y_0) < n$  then by property 1 of index there exists no equivariant map from  $X$  to  $Y \setminus Y_0$ , hence  $(f|_X)^{-1}(Y_0) \neq \emptyset$  (in this case we don't need the assumption 2).

The theorem follows also from the following result.



**Theorem 5.2.** *Let  $X_0$  be a free  $G$ -space,  $i : X_0 \subset X$ . Let  $K$  be a free  $G$ -space and  $f : X \rightarrow K$  is a map equivariant on  $X_0$ . Assume that  $\text{ind}^G X_0 = d$  and that  $X_0$  is  $d$ - $\mathbb{Z}$ -c.t.-subspace of  $X$ . Then  $\text{ind}^G K \geq d + 1$ .*

*If in addition  $K$  is a connected closed orientable topological  $(d + 1)$ -dimensional manifold or a pseudomanifold then for any  $y \in K$  at least one of the sets  $f^{-1}(gy)$  for some  $g \in G$  depending on  $y$  is nonempty.*

*Proof.* The map  $f \circ i : X \rightarrow K$  is equivariant, so  $\text{ind}^G K \geq \text{ind}^G X = d$ . Since  $(f \circ i)^* = i^* \circ f^* = 0$  in dimension  $d$ , it follows from property 8 of index that  $\text{ind}^G K \neq d$ . Therefore  $\text{ind}^G K \geq d + 1$ .

When  $K$  is a manifold we argue by contradiction. Let  $y \in K$  be a point such that  $f^{-1}(Gy) = \emptyset$  where  $Gy$  is the orbit of the point  $y$ . Then  $f$  maps  $X$  to  $K \setminus Gy$  and  $f \circ i : X_0 \rightarrow K \setminus Gy$  is equivariant. Applying the first statement we obtain that  $\text{ind}^G (K \setminus Gy) \geq d + 1$ . On the other hand  $K \setminus Gy$  is an open manifold, hence  $H^j(K \setminus Gy; \mathbb{Z}) = 0$  for  $j \geq d + 1$ , and from property 7 of index we obtain  $\text{ind}^G (K \setminus Gy) < d + 1$ . (Also  $H^{d+1}(K \setminus Gy; \mathbb{Z}) = 0$  contradicts with  $\text{ind}^G (K \setminus Gy) = d + 1$  by property 8.)  $\square$

**Definition 5.2.** *Let  $Y$  be a  $G$ -space. A point  $y \in Y$  is a fixed point of the action if  $gy = y \ \forall g \in G$ . Denote the set of fixed points by  $Y^G$ . We say that the action of  $G$  on  $Y$  is semifree if  $Y \setminus Y^G \neq \emptyset$  and  $Y^G \neq \emptyset$  and  $G$  acts freely on  $Y \setminus Y^G$ .*

Assume that  $Y$  is a semifree  $G$ -space and  $f : X \rightarrow Y$  a continuous map. In this case directly from theorem 5.1 we obtain:

**Theorem 5.3.** *Let  $Y$  is a semifree  $G$ -space,  $f : X \rightarrow Y$  a continuous map.*

*Assume that*

- 1)  $n = \text{ind}^G X_0 = \text{ind}^G (Y \setminus Y^G)$ ,
- 2)  $X_0$  is  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}$ .
- 3)  $f|_{X_0} : X \rightarrow Y$  is equivariant.

*Then  $f^{-1}(Y^G) \neq \emptyset$ .*

We can apply this result in the case when  $X = M$  is a manifold and  $X_0 = \partial M$  is its boundary.

**Theorem 5.4.** *Let  $M^n$  be a compact connected orientable manifold (or a pseudomanifold) with the connected boundary  $\partial M$ , and assume that  $G$  can act freely on  $\partial M$ . Consider a continuous mapping  $f : M \rightarrow \mathbb{R}^n$  such that  $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$  is an equivariant map, where  $\mathbb{R}^n$  is considered as a semifree  $G$ -space with the unique fixed point  $0 \in \mathbb{R}^n$ , the origin. If  $\text{ind}^G \partial M = n - 1$  then the zero set  $Z_f = f^{-1}(0)$  is not empty.*

Here we consider any semifree action of  $G$  on  $\mathbb{R}^n$  with the unique fixed point  $0 \in \mathbb{R}^n$ , the origin. Such an action exists since we assume that  $G$  can act freely on  $S^{n-1}$ . For example we can take the action which is obtained by linearity from the  $G$ -action on  $S^{n-1}$ .

As a partial case of the previous assertion we obtain:

**Corollary 5.1.** *Let  $M^n$  be a compact connected orientable manifold (or a pseudomanifold) with the boundary  $\partial M$  which is homeomorphic to the sphere  $\mathbb{S}^{n-1}$ , and assume that  $G$  can act freely on  $\partial M \approx \mathbb{S}^{n-1}$ . Consider a continuous mapping  $f : M \rightarrow \mathbb{R}^n$  such that  $f|_{\partial M} : \partial M \rightarrow \mathbb{R}^n$  is an equivariant map, where  $\mathbb{R}^n$  is considered as a semifree  $G$ -space with the unique fixed point  $0 \in \mathbb{R}^n$ , the origin. Then the zero set  $Z_f = f^{-1}(0)$  is not empty.*

This follows also from

**Proposition 5.1.** *If there is an embedding  $i : \mathbb{S}^{d-1} \rightarrow X$  such that*

$$\text{Im } i^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset,$$

*where  $i^* : H^{d-1}(X; \mathbb{Z}) \rightarrow H^{d-1}(\mathbb{S}^{d-1}; \mathbb{Z})$ , and  $f : X \rightarrow \mathbb{R}^d$  a continuous map such that  $f|_{\mathbb{S}^{d-1}} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$  is equivariant, then  $0 \in f(X)$ .*

Actually a more general assertion holds

**Theorem 5.5.** *Assume that there is a map  $j : \mathbb{S}^{d-1} \rightarrow X$  such that*

$$\text{Im } j^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset,$$

*where  $j^* : H^{d-1}(X; \mathbb{Z}) \rightarrow H^{d-1}(\mathbb{S}^{d-1}; \mathbb{Z})$  is induced by  $j$ , and let  $f : X \rightarrow \mathbb{R}^d$  be a continuous map such that  $f \circ j : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$  is equivariant. Then  $0 \in f(X)$ .*

*Proof.* We argue by a contradiction. If  $0 \notin f(X)$ , then  $f \circ j : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d \setminus 0$  is an equivariant map, hence its degree equals 1 modulo  $|G|$  (see f.e. [9]), but this contradicts with the assumption  $\text{Im } j^* \cap \{k \in \mathbb{Z} \mid k \equiv 1 \pmod{|G|}\} = \emptyset$ .  $\square$

For more conclusions in the case when  $X_0$  is a sphere see the Appendix.

## 6 Bourgin–Yang type theorems

**Definition 6.1.** *Let  $X$  be a space and  $X_0$  its subspace which is a  $G$ -space. A camomile  $C$  is a  $G$ -space for which there is an embedding  $X \subset C$  such that  $C = GX$ , induced embedding  $X_0 \subset C$  is equivariant, the action of  $G$  on  $C \setminus X_0$  is free, and  $C \setminus X_0 = \bigcup_{g \in G} g(X \setminus X_0)$ .*

**Example 6.1.** If  $X$  is a cone over  $X_0$ , i.e.  $X = X_0 * \text{pt}$ , then  $C = X_0 * G$ .

Let  $Y$  be a  $G$ -space and  $Y_0$  its invariant subspace such that the  $G$ -action on  $Y \setminus Y_0$  is free. From the definition of camomile we easily obtain the following assertion.

**Theorem 6.1.** *There exists  $f : X \rightarrow Y$  equivariant on  $X_0$  and such that  $f^{-1}(Y_0) = \emptyset$  if and only if there exists an equivariant map  $C \rightarrow Y \setminus Y_0$  where  $C$  is the camomile associated with the embedding  $X_0 \subset X$  of the  $G$ -space  $X_0$  into  $X$ .*

**Theorem 6.2.** *Let  $X_0$  be an  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}$  such that  $\text{ind}^G X_0 = n$ . Then  $\text{ind}^G C = n + 1$ .*

*Proof.* Since the inclusion  $X_0 \subset C$  is equivariant, we have from property 1 that  $\text{ind}^G C \geq n$  and from property 8 obtain that  $\text{ind}^G C \geq n + 1$ . By property 3 there exists an invariant neighborhood of  $X_0$  in  $C$  of index  $n$ . A complement of this neighborhood is a  $G$ -space that can be mapped equivariantly to  $G$ , so its index equals zero. Hence from property 2 we obtain that  $\text{ind}^G C \leq n + 1$ .  $\square$

Now we show how to construct a camomile in the case when  $X$  is a finite-dimensional compact space and  $X_0$  its closed subspace (so  $X_0$  is a compacta also).

By Mostow theorem [13] we can equivariantly embed  $X_0$  into finite-dimensional Euclidean  $G$ -space  $V$ . By Tietze lemma we can extend this embedding to the map  $\varphi : X \rightarrow V$ . If  $\dim X = k$  then using Nöbeling–Pontrjagin theorem we can embed  $X$  into the unit sphere  $S^{2k+1} \subset \mathbb{R}^{2k+2}$ . Denote this embedding by  $\psi : X \rightarrow \mathbb{R}^{2k+2}$ . Define a real-valued function  $h : X \rightarrow \mathbb{R}$  as  $h(x) = \rho(x, X_0)$ , the distance between a point  $x$  and  $X_0$ . This function takes zero values on  $X_0$  and is positive on  $X \setminus X_0$ . Define  $\eta : X \rightarrow \mathbb{R}^{2k+2}$  as  $\eta(x) = h(x)\psi(x)$ . Then  $\zeta : X \rightarrow V \oplus \mathbb{R}^{2k+2}$ ,  $\zeta(x) = (\varphi(x), \eta(x))$ , is an embedding. We will consider  $V \oplus \mathbb{R}^{2k+2}$  as a Euclidean  $G$ -space ( $G$  acts trivially on  $\mathbb{R}^{2k+2}$ ). Then  $\zeta$  is the embedding which is equivariant on  $X_0$ . Finally put  $W = V \oplus \mathbb{R}^{2k+2} \oplus \mathbb{R}[G]$  where  $\mathbb{R}[G]$  is the group ring considered as Euclidean space of dimension  $|G|$ , the order of  $G$ . The group acts on  $\mathbb{R}[G]$  by left multiplication and it is convenient to denote basis vectors as elements of the group  $G$ , so  $\mathbb{R}[G] = \bigoplus_{g \in G} \mathbb{R} \cdot g$ . Now we define an embedding  $\mu : X \rightarrow V \oplus \mathbb{R}^{2k+2} \oplus \mathbb{R} \cdot e \subset W$  where  $e \in G$  is the unit of  $G$  by the formula  $(\varphi(x), h(x)\psi(x), h(x) \cdot e)$ . Then  $\mu : X \rightarrow W$  is equivariant on  $X_0$  and  $C = G\mu(X)$  is a camomile.

In fact the same construction of the camomile is valid for finite-dimensional separable metric space  $X$  and closed subspace  $X_0$  (with  $G$ -action).

Camomile is convenient for proving results of Bourgin–Yang type.

**Theorem 6.3.** *Assume that  $Y$  is a  $G$ -space,  $Y_0$  its invariant closed subspace such that the action on  $Y \setminus Y_0$  is free, and  $f : X \rightarrow Y$  a continuous map. If*

- 1)  $n = \text{ind}^G X_0 > \text{ind}^G (Y \setminus Y_0)$ ,
  - 2)  $X_0$  is an  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}$ ,
  - 3)  $f|_{X_0} : X_0 \rightarrow Y$  is equivariant,
- then  $\dim f^{-1}(Y_0) \geq n - \text{ind}^G (Y \setminus Y_0)$ .*

*Proof.* We have  $\text{ind}^G C = n + 1$ , where  $C$  is the camomile. Denote by  $h : C \rightarrow Y$  the equivariant extension of  $f$ . Then  $\text{ind}^G h^{-1}Y_0 \geq \text{ind}^G C - \text{ind}^G (Y \setminus Y_0) - 1 = n - \text{ind}^G (Y \setminus Y_0)$ , hence  $\dim h^{-1}Y_0 \geq n - \text{ind}^G (Y \setminus Y_0)$ . Since  $h^{-1}Y_0 = \bigcup_{g \in G} g \cdot f^{-1}(Y_0)$  and  $\dim g \cdot f^{-1}(Y_0) = \dim f^{-1}(Y_0)$  for any  $g \in G$ , we are done.  $\square$

Since a free  $G$ -space is a free space in respect to any subgroup we have analogs of the above results in which  $\text{ind}^G(\cdot)$  is replaced by  $\text{ind}_p(\cdot)$  where  $p = |H|$  is a prime and  $H$  is some subgroup of  $G$ . For example we have the following result:

**Theorem 6.4.** *Let  $Y$  be a  $G$ -space,  $Y_0$  its invariant closed subspace such that the action on  $Y \setminus Y_0$  is free, and  $f : X \rightarrow Y$  a continuous map. Let  $H = \mathbb{Z}_p$ ,  $p$  is a prime, be a subgroup of  $G$ . Assume that*

- 1)  $n = \text{ind}_p X_0 \geq \text{ind}_p(Y \setminus Y_0)$ ,
- 2)  $X_0$  is an  $n$ -c.t.-subspace of  $X$  over  $\mathbb{Z}_p$ ,
- 3)  $f|_{X_0} : X \rightarrow Y$  is equivariant.

*Then  $f^{-1}(Y_0) \neq \emptyset$ .*

*If  $n = \text{ind}_p X > \text{ind}_p(Y \setminus Y_0)$  then  $\dim f^{-1}(Y_0) \geq n - \text{ind}_p(Y \setminus Y_0)$ .*

## 7 Tucker type lemmas for bounded spaces

Consider the case of simplicial complex  $X$  and its subcomplex  $X_0$ . We assume that  $G$  acts freely and simplicial on  $X_0$ .

**Theorem 7.1.** *Assume that  $\text{ind}^G X = n - 1$  and that  $X_0$  is an  $(n - 1)$ -c.t.-subspace of  $X$  over  $\mathbb{Z}$ . Then for any  $(G, n)$ -labeling of the vertex set of an arbitrary triangulation of  $X$  which is equivariant on  $X_0$  there exists a complementary edge.*

*Proof.* We argue by contradiction. A  $(G, n)$ -labeling of the vertex set of a triangulation of  $X$  without complementary edges provides a map  $\psi : X \rightarrow J^n(G)$ , and this map is equivariant on  $X$  since our  $(G, n)$ -labeling is equivariant on  $X$ . Since  $i^*$  is trivial in dimension  $n - 1$ , where  $i : X_0 \subset X$  is the inclusion, we see that  $(\psi|_X)^* : H^{n-1}(J^n(G); \mathbb{Z}) \rightarrow H^{n-1}(X; \mathbb{Z})$  is trivial, and we obtain a contradiction with property 8 of cohomological index, because  $\text{ind}^G X = n - 1 = \text{ind}^G J^n(G)$ .  $\square$

As a partial case we obtain:

**Theorem 7.2.** *Let  $M^n$  be a compact PL manifold with boundary. Suppose that  $\partial M$  is homeomorphic to the sphere  $\mathbb{S}^{n-1}$  and there exists a free PL action of a group  $G$  on  $\partial M \approx \mathbb{S}^{n-1}$ . Then for any  $(G, n)$ -labeling of the vertex set of an arbitrary triangulation of  $M$  that is an equivariant on the boundary there exists a complementary edge.*

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O. R. Musin, University of Texas Rio Grande Valley, Brownsville, TX, 78520  
*E-mail address:* oleg.musin@utrgv.edu

A. Yu. Volovikov, Department of Higher Mathematics, MTU MIREA, Vernadskogo av., 78, Moscow, 119454, Russia  
*E-mail address:* a\_volov@list.ru